Problem: Consider we have a relative depth map $f : [0, h] \times [0, w] \to \mathbb{R}$, as well as n points $(a_1, b_1), \ldots, (a_n, b_n)$ with absolute depth values $d_1, \ldots, d_n$. Letting

$$d_{rel} = [f(a_1, b_1), f(a_2, b_2), \ldots, f(a_n, b_n)]$$

, the goal is to find some scaling factor $c \in \mathbb{R}$ to minimize the distance between vectors

$$d_{rel,scal} = [cf(a_1, b_1), cf(a_2, b_2), \ldots, cf(a_n, b_n)]$$

and

$$d_{abs} = [d_1, d_2, \ldots, d_n]$$

The first is the scaled depth predictions of the relative depth map and the second is the absolute depth values. We find the min via differentiation:

$$\begin{aligned}
&\min_{c \in \mathbb{R}} \|d_{rel,scal} - d_{abs}\|_2 \\
= &\min_{c \in \mathbb{R}} \|d_{rel,scal} - d_{abs}\|_2^2 \\
= &\min_{c \in \mathbb{R}} \sum_{i \in [n]} (cf(a_i, b_i) - d_i)^2 \\
&\text{Now we differentiate w.r.t. } c \text{ to find the min value} \\
&\frac{\partial f}{\partial c} = 0 \\
\iff &\sum_{i \in [n]} 2(c_{opt} f(a_i, b_i) - d_i) f(a_i, b_i) = 0 \\
\iff &\sum_{i \in [n]} 2d_i f(a_i, b_i) = \sum_{i \in [n]} 2c_{opt} f(a_i, b_i)^2 \\
&\text{Note that all coefficients of } c_{opt} \text{ are nonnegative,} \\
&\text{so the second derivative is nonegative and thus we have a minima} \\
\iff &\frac{\sum_{i \in [n]} d_i f(a_i, b_i)}{\sum_{i \in [n]} f(a_i, b_i)^2} = c_{opt} \\
\iff &\frac{d_{abs} \cdot d_{rel}}{\|d_{rel}\|_2^2} = c_{opt}
\end{aligned}$$

We have found that the optimal value of c is the dot product of the two original vectors over the squared magnitude of the relative depth vector. For the interested reader, this is the minimum distance between point d_{abs} and a line passing through the origin and d_{rel} in $\mathbb{R}^n$ (here this line is parameterized by the scaling constant c)